Induction

http://www.picshag.com/recursive-painting.html

Discrete Structures (CS 173)

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Lectures based on Derek Hoiem, University of Illinois

Last lecture: graphs and 2-way bounds

- Terminology for graph connectivity
  - Walk, path, cycle, acyclic, closed, Euler circuit, distance, diameter, connected components
- Graph coloring and how to apply it
- How to use two-way bounding in a variety of settings

This lecture (and next): Induction

- What is Mathematical Induction
- Examples
- Non-examples

History of Induction

- First ‘conscious’ use of the principle of induction was by Blaise Pascal (1623-1662)
- Historians argue about whether it was used by ancient philosophers, other civilizations
  - Counter-argument is the notion of variables wasn’t formalized until after Al-Khwarzimi.
  - But Plato uses an inductive argument in *Parmenides.*

Reference: Plato: Parmenides 149a7-c3. A Proof by Complete Induction?Author(s): F. AcerbiSource: Archive for History of Exact Sciences, Vol. 55, No. 1 (August 2000), pp. 57-76 Published by: SpringerStable URL: http://www.jstor.org/stable/41134098

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- Suppose domino k falls. Then domino k+1 falls.

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- Suppose domino k falls. Then domino k+1 falls.

$$\text{fall}(k) \rightarrow \text{fall}(k+1)$$

- The first domino falls

$$\text{fall}(1) = T$$

Induction

Inductive hypothesis: Suppose domino k falls.

Inductive conclusion: Domino k+1 falls.

Base case: The first domino falls.

Simple math example

Claim: $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ for all positive integers n.

Basic structure of induction proof

Claim: $P(n)$ for all $n = 0, 1, 2, 3 \ldots$

To prove:

1. Base: $P(0)$ is true.
2. Inductive step: $\underset{\substack{\nwarrow \\ \text{Inductive} \\ \text{hypothesis}}}{P(k)} \Longrightarrow \underset{\substack{\nwarrow \\ \text{Inductive} \\ \text{conclusion}}}{P(k+1)} \Bigg\}$ Weak Induction

or $\quad (P(0), P(1), \ldots P(k)) \Longrightarrow P(k+1) \Bigg\}$ Strong Induction

Principle of Induction (on Natural Numbers)

If $\forall i, n \in \mathbb{N}\ (\wedge_{i<n} P(i) \Rightarrow P(n))$ then $\forall n \in \mathbb{N}, P(n)$

What about the base case?

Infinite Descending chain

A *descending chain* is a sequence of elements $a_1, a_{2,}\ \ldots, a_n$ such that $a_1 > a_2\ > \cdots > a_n$

An *infinite descending chain* is an infinite sequence of elements a_1, a_2, … such that $a_1 > a_2\ > \cdots >$ …

Note: < may be a partial order (need not be a total order)!

Well-founded Sets

If a set has no infinite descending chains, it is called *well-founded.*

Induction and Well-foundedness

- If there are no infinite descending chains defined by a relation $\prec$ then the principle of Induction holds over the relation $\prec$

Minimal Elements

Definition. $x \in A$ is a *minimal element of* A *iff* $\forall y \in A, y \nless x$.

Well-foundedness

Proposition. The set S has no infinite descending chains if and only if every subset of S has a minimal element.

Another example

Claim: For $n = 2, 3, 4, \ldots$, $\sum_{i=2}^{n} i2^{i} = (n-1)2^{n+1}$.

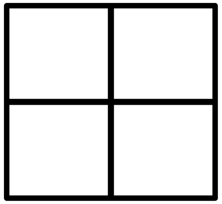
Number theory example

Claim: For any natural integer n, $n^3 - n$ is divisible by 3.

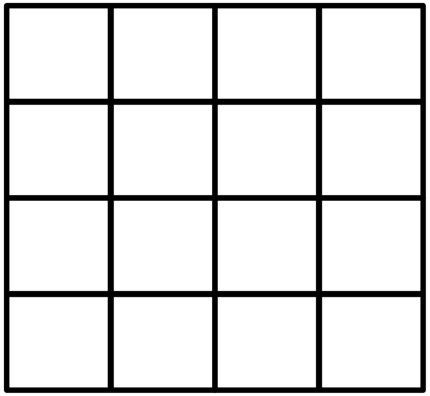
Geometrical example

Claim: For any positive integer n, a $2^n \times 2^n$ checkerboard with one square removed can be tiled using triminos.

right trimino



$2^1 \times 2^1$



$2^2 \times 2^2$

Top Down Induction

- Proof: Let n be an arbitrary non-negative integer. Assume that for any non-negative integer $k < n$, the 2k ×2k grid with any square removed can be tiled

One more example

Claim: $\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$ for all positive integers n.

Binary Numbers

Theorem: Every natural number can be written as the sum of distinct powers of 2. (i.e., can be written in binary).

Proof (Top down induction):

1. $n = 0$. Trivial. $n = \sum_{i \in \emptyset} i = 0$
2. Let $n > 0$. Let k be the largest integer such that $2^k < n$, and let $m = n - 2^k$.

Now $m < 2^{k+1} - 2^k$. Because $0 \leq m \leq n$, by the inductive hypothesis, m can be written as a sum of distinct powers of 2. Moreover, in this summation, each power is at most m, therefore it is less than 2^k.

Thus $n = m + 2^k$ is the sum of distinct powers of 2.

Binary Numbers

Theorem: Every natural number can be written as the sum of distinct powers of 2. (i.e., can be written in binary).

Proof (Bottom up induction):

1. $n = 0$. Trivial. $n = \sum_{i \in \emptyset} i = 0$
2. Let $n > 0$. Let $m = \left\lfloor \frac{n}{2} \right\rfloor$.

Because $0 \leq m \leq n$, by the inductive hypothesis, m can be written as a sum of distinct powers of 2.

Thus $2m$ can be written as a sum of distinct powers of 2 (add one to each exponent), each of the powers is greater than 2^0.

If n is even, $n = 2m$. Otherwise $n = 2m + 1 = 2m + 2^0$.

Things to remember

- Induction works (only) on well-founded sets.
- Induction requires demonstrating a **base case** and an **inductive step**
- Inductive step usually involves showing that $P(k) \rightarrow P(k+1)$ or $P(\text{all}\, n \text{ up to } k) \rightarrow P(k+1)$
  - Typically, this requires writing $P(k+1)$ in terms of $P(k)$

False Induction

Proposition: In any set of n horses, all horses have the same color.

Proof by Induction of the size of the set.

1. For $n = 1$, all horses have the same color.

2. *(Inductive hypothesis).* Suppose the property holds for all $k \leq n$.

Now consider $n = k + 1$. Divide the set of horses into two sets S_1, S_2 of size $n_1, n_2 \leq n$, so that both sets contain a horse h. By inductive hypothesis, all horses in S_1 $(\text{resp.}\, S_2)$ have the same color. But they contain horse h, so all horses in both sets have the same color as h.

QED

Zeno’s Paradox: Rabbit and Tortoise

Consider Rabbit, who runs at 1 km/min and Tortoise, who runs at 0.1 km/min. They start a race. Rabbit naps for 10 min.

Claim: Rabbit can never catch up with Tortoise.

Proof by induction: To catch up Rabbit must first reach where Tortoise is.

1. Base case: Rabbit goes 1 km, Tortoise is at 1.1 km. So Rabbit has not caught up.
2. Assume Rabbit has not caught up at the k^{th} *place Tortoise is (δ ahead).* Rabbit catches up with Tortoise. But now Tortoise is 0.1 δ. . So Rabbit has not caught up.

Conclusion: Rabbit never catches up. QED.

What is wrong with the induction?

Next class

- Induction with graphs, stamps, and games